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Hamiltonian BRST Quantization of the Conformal String

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Abstract: We present a new formulation of the tensionless string ($T = 0$) where the space-time conformal symmetry is manifest. Using a Hamiltonian BRST scheme we quantize this *Conformal String* and find that it has critical dimension $D = 2$. This is in keeping with our classical result that the model describes massless particles in this dimension. It is also consistent with our previous results which indicate that quantized conformally symmetric tensionless strings describe a topological phase away from $D = 2$.

We reach our result by demanding nilpotency of the BRST charge and consistency with the Jacobi identities. The derivation is presented in two different ways: in operator language and using mode expansions.

Careful attention is payed to regularization, a crucial ingredient in our calculations.

Introduction

The high-energy limit of strings has been studied with regard to scattering [1]-[7] as well to the high-temperature behaviour [8]-[11], but it is far from fully understood. Open problems are the understanding of the high-energy symmetries of Gross [5] and Moore [6], and the relation to the conjectured "topological phase" of general covariance [12],[13].

The zero tension limit ($T \rightarrow 0$) of strings and superstrings, [14]-[25] provide a possible high energy limit of the corresponding tensile ($T \neq 0$) models[1]. They are also interesting in their own right, since they provide new, albeit somewhat degenerate, string models. In addition their quantization is sufficiently different from the ($T \neq 0$) models to provide new insights into the quantization of extended objects.

In a previous article, [15], the condition under which the space-time conformal symmetry of the bosonic tensionless string survives quantization was investigated. The surprising conclusion is that this symmetry holds good at the quantum level essentially only if the physical states of the theory are space-time diffeomorphism singlets, indicating that the theory describes a topological string phase.

The treatment in [15] is based on an action with only the Poincaré subgroup of the space-time conformal symmetry manifest. Furthermore the quantization is carried out in a light-cone gauge describing only physical degrees of freedom. Thus, all of the space-time conformal symmetry has to be explicitly checked. In view of the surprising outcome it is important to corroborate the results in [15] using other methods. A first step in this direction is to identify the obstructions to quantization using different quantization schemes.

In this paper we present a Hamiltonian BRST quantization of the $T \rightarrow 0$ limit of the bosonic string, starting from *the Conformal String*, (named in analogy to the conformal particle in [30]), a $D+2$ dimensional formulation which is classically equivalent to that used in [15], but where the space-time conformal symmetry *is manifest*[2]. We find that this theory has critical dimension $D = 2$, a result which is consistent with [15] where the $D-2$ transversal degrees of freedom are the basic objects and where hence $D \neq 2$

[1]See, e.g., [18] [15].

[2]In $D = 2$ only invariance under the finite dimensional möbius subgroup of the conformal group is manifest.

from the outset. It is also in keeping with the classical results presented in the present paper that our model describes massless particles in $D = 2$. As was shown in [15], the conformal invariance for massless particles in $D > 2$ survives quantization and one would expect that to be true in $D = 2$ also[3]. Finally, the space-time conformal group is infinite-dimensional in $D = 2$ which is also expected to give good quantum behaviour.

The content of the paper is as follows:

In Section 1 we present the classical Conformal String theory, (the $D + 2$ dimensional action with manifest conformal symmetry), its symmetries, equations of motion and some of their consequences. In particular we discuss the relation to other formulations, the classical picture as a set of conformal particles [30] obeying a constraint, the Hamiltonian description in terms of the classical constraints and their algebra and the classical BRST charge.

Section 2 is the main part of our paper and contains the quantum theory. It starts out with a discusssion of the vacuum (2.1-2). In many ways the $T = 0$ string behaves like a collection of particles, and the vacuum we find appropriate is indeed annihilated by the momentum operators. Starting from this requirement we find the full vacuum which accomodates the existence of ghosts and should allow for finite inner products in analogy to [27]. The key problem in our calculations is to keep track of possible divergencies. The method for doing this is to introduce a regularized delta function in the canonical commutation-relations and then to choose a particular "physical" ordering of the coordinate and momentum operators in the calculations involving the composite expressions for the constraints. This prescription and its application in investigating the nilpotency of the quantum BRST charge is contained in (2.3-4), concluding with the discovery of the critical dimension $D = 2$.

In (2.5) we set out anew with slightly different approach. Here we use a mode expansion of the operators and constraints and regulate infinite sums rather than delta functions. We discuss the central extensions of the quantum constraint algebra, introduce "extended constraints", i.e., include the ghost sector in the constraints, and derive the consequences of the central

[3] This is independent of the usual difficulties with masslessness in $D = 2$.

extensions for the nilpotency of the BRST charge in an economic way. Here the effect on the Jacobi identities is also discussed. We discover that the Jacobi identities hold if and only if $D = 2$ and that the BRST charge is then nilpotent.

In Section 3 we reexamine the classical theory in the critical dimension $D = 2$, discuss our results, mentioning how we believe that the present results can be reconciled with the results of [15] away from $D = 2$, and point to some future topics for investigation.

In Appendices A and B we have collected some explicit calculations along with a presentation of how the critical dimension $D = 26$ for $T \neq 0$ strings is derived using our methods.

1 The classical theory

The "Conformal String Theory" we consider is given by the action

$$S = \int d^2\xi \left(V^\alpha V^\beta \mathcal{D}_\alpha X^M \mathcal{D}_\beta X^N \eta_{MN} + \Phi X^2 \right) \tag{1}$$

where $X^M(\xi), \quad M = 0, ..., D+1$ is an embedding of the world sheet, coordinatized by $(\xi^\alpha) = (\xi^0, \xi^1) \equiv (\tau, \sigma)$, into the target space with metric

$$\eta_{MN} = \begin{pmatrix} \eta_{mn} & 0 & 0 \\ 0...0 & 1 & 0 \\ 0...0 & 0 & -1 \end{pmatrix}. \tag{2}$$

Here $\eta_{mn}, \quad m = 0, ..., D-1$, is the metric in D-dimensional Minkowski space, which shows that we have written the theory in a $D+2$ dimensional space with signature $(-+++......+-)$. Furthermore V^α is a contravariant vector-density (whose transformation properties will be given below) and the scale-covariant derivatives $\mathcal{D}_\alpha$ are given by

$$\mathcal{D}_\alpha \equiv \partial_\alpha + W_\alpha \tag{3}$$

with W_α being the gauge field for scale transformations. Finally $X^2 \equiv X^M X^N \eta_{MN}$ and Φ is a scalar density Lagrange multiplier field that restricts

the theory to the $D+2$ dimensional light cone. The model is reminiscent of the conformal particle [30], hence its name.

The action (1) is a $D+2$ dimensional version of the action first used in [18] and subsequently employed in investigations of the $T \to 0$ limit of strings [19, 20, 14, 15]. Just like the slightly different action in [17] the space-time conformal symmetry has been made manifest by adding one time-like and one space-like dimension. In Hamiltonian form this theory was also treated in [26].

The symmetries of the action (1) are two dimensional (world sheet) diffeomorphisms, local $D+2$ scale transformations[4], an "additional" local two dimensional symmetry and global $(D+2)$-dimensional rotations. Explicitly, they are given by:

(i) Diffeomorphisms ($\epsilon = \epsilon(\xi)$):

$$\begin{aligned} \delta_\epsilon X^M &= \epsilon^\alpha \partial_\alpha X^M \equiv \epsilon \cdot \partial X^M \\ \delta_\epsilon V^\alpha &= \epsilon \cdot \partial V^\alpha - V \cdot \partial \epsilon^\alpha + \tfrac{1}{2}(\partial \cdot \epsilon) V^\alpha \\ \delta_\epsilon W_\alpha &= \epsilon \cdot \partial W_\alpha + W_\beta \partial_\alpha \epsilon^\beta \\ \delta_\epsilon \Phi &= \partial_\alpha(\epsilon^\alpha \Phi) \end{aligned} \tag{4}$$

(ii) Scale transformations ($\lambda = \lambda(\xi)$):

$$\begin{aligned} \delta_\lambda X^M &= \lambda X^M \\ \delta_\lambda V^\alpha &= -\lambda V^\alpha \\ \delta_\lambda W_\alpha &= -\partial_\alpha \lambda \\ \delta_\lambda \Phi &= -2\lambda\Phi \end{aligned} \tag{5}$$

(iii) Additional symmetry ($\Xi_\alpha = \Xi_\alpha(\xi)$):

$$\begin{aligned} \delta_\Xi X^M &= 0 \\ \delta_\Xi V^\alpha &= 0 \\ \delta_\Xi W_\alpha &= -\Xi_\alpha \\ \delta_\Xi \Phi &= 2(V \cdot W)(V \cdot \Xi) - \partial_\alpha(V^\alpha V \cdot \Xi) \end{aligned}$$

[4] Not to be confused with the D-dimensional dilatations of the conformal group, which are included in the $(D+2)$-dimensional rotations.

$$(6)$$

(iv) Rotations ($\Lambda_{MN} \equiv \Lambda_{[MN]}$):

$$\begin{aligned}
\delta_\Lambda X^M &= \Lambda^M_N X^N \\
\delta_\Lambda V^\alpha &= 0 \\
\delta_\Lambda W_\alpha &= 0 \\
\delta_\Lambda \Phi &= 0
\end{aligned} \tag{7}$$

The field equations that result from the action (1) are:

$$\begin{aligned}
\delta V^\alpha : &\qquad \mathcal{D}_\alpha X^M V^\beta \mathcal{D}_\beta X_M = 0 \\
\delta X^M : &\qquad \mathcal{D}_\alpha (V^\alpha V^\beta \mathcal{D}_\beta X_M) - \Phi X_M = 0 \\
\delta W_\alpha : &\qquad V^\alpha V^\beta X^M \mathcal{D}_\beta X_M = 0 \\
\delta \Phi : &\qquad X^2 = 0,
\end{aligned} \tag{8}$$

where, in the second line, $\mathcal{D}_\alpha(\ )^\alpha = (\partial_\alpha - W_\alpha)(\ )^\alpha$, the change in sign being due to the scaling property of the term it acts on.

In a reparametrization gauge $V^\alpha = (E^{-\frac{1}{2}}(\tau), 0)$ the equations (8) become

$$\begin{aligned}
X^2 &= 0 \\
\dot{X}^2 &= 0 \\
\ddot{X}^M &= \frac{\dot{E}}{E}\dot{X}^M + \left(\tilde{\Phi} E^2 + W^2 + \frac{\dot{E}}{E} W - \dot{W}\right) X^M \\
\dot{X}^M X'_M &= 0
\end{aligned} \tag{9}$$

where $\tilde{\Phi} \equiv \Phi E^{-1}$ and $W \equiv W_0$. Here dot denotes τ- and prime denotes σ- derivatives. For fixed σ, the three first equations in (9) are precisely the equations of motion for the conformal particle with action [30]:

$$S = \int d\tau \left(E^{-1} (\dot{X} + W)^2 + E \Phi X^2 \right). \tag{10}$$

Hence the conformal string may be viewed (in this gauge) as a collection of conformal particles, one at each σ, subject to a constraint, (the last equation in (9)). Also in the Minkowski space formulation of the zero tension limit of the bosonic, spinning and superstrings there are similar gauge choices where massless particle, spinning particle and superparticle equations may be recognized.

Integrating out W_α and fixing a scaling gauge, the action (1) can be reduced to the space-time action employed in, e.g., [15],

$$S = \int d^2\xi V^\alpha \partial_\alpha X^m V^\beta \partial_\beta X_m, \tag{11}$$

where the Minkowski metric η_{mn} is used in the X summation.

In the remaining part of this letter we will be interested in the Hamiltonian form of the theory. It is given by the Hamiltonian

$$H = \lambda_i \phi^i, \quad i = -1, 0, 1, L \tag{12}$$

where λ_i are Lagrange multiplier fields[5] and the constraints ϕ^i are given by:

$$\begin{array}{rcl} \phi^{-1} & = & P^2 \\ \phi^0 & = & P_M X^M \\ \phi^1 & = & X^2 \\ \phi^L & = & P_M X'^M, \end{array} \tag{13}$$

with X^M and P_N fulfilling the usual canonical relations

$$\left\{X^M(\sigma), P_N(\sigma')\right\} = \delta^M_N \delta(\sigma - \sigma'). \tag{14}$$

These constraints are all first class and form the following algebra,

$$\begin{array}{rcl} \left\{\phi^1(\sigma), \phi^{-1}(\sigma')\right\} & = & 2\left(\phi^0(\sigma) + \phi^0(\sigma')\right)\delta(\sigma - \sigma') \\ \left\{\phi^1(\sigma), \phi^0(\sigma')\right\} & = & \left(\phi^1(\sigma) + \phi^1(\sigma')\right)\delta(\sigma - \sigma') \\ \left\{\phi^0(\sigma), \phi^{-1}(\sigma')\right\} & = & \left(\phi^{-1}(\sigma) + \phi^{-1}(\sigma')\right)\delta(\sigma - \sigma') \\ \left\{\phi^L(\sigma), \phi^L(\sigma')\right\} & = & \left(\phi^L(\sigma) + \phi^L(\xi')\right)\delta'(\xi - \xi') \\ \left\{\phi^1(\sigma), \phi^L(\sigma')\right\} & = & \left(-\phi^1(\sigma) + \phi^1(\sigma')\right)\delta'(\sigma - \sigma') \\ \left\{\phi^0(\sigma), \phi^L(\sigma')\right\} & = & \phi^0(\sigma)\delta'(\sigma - \sigma') \\ \left\{\phi^{-1}(\sigma), \phi^L(\sigma')\right\} & = & \left(\phi^{-1}(\sigma) + \phi^{-1}(\sigma')\right)\delta'(\sigma - \sigma'). \end{array} \tag{15}$$

All other Poisson brackets are zero. Here ϕ^L generates a Virasoro algebra and ϕ^i transform under this algebra with conformal spin $1-i$. The whole algebra

[5]The Lagrange multipliers correspond to the fields V, W and Φ in the Lagrange formulation.

is a semi direct product between the Virasoro algebra and a $SU(1,1)$ Kač-Moody algebra, both without central extensions. Note that the subalgebra formed by ϕ^L and ϕ^{-1} is isomorphic to the gauge algebra in the Minkowski formulation of the tensionless string [15]. With the structure constants of the constraint algebra at hand we may write down the classical Hamiltonian BRST charge $\mathcal{Q}$ [31, 32]:

$$\mathcal{Q} = \int d\sigma \left(\phi^i c^i + 4b^0 c^{-1} c^1 + 2b^1 c^0 c^1 + 2b^{-1} c^{-1} c^0 + \partial b^1 c^L c^1 \right.$$
$$\left. + b^{-1} \partial c^{-1} c^L + b^0 \partial c^0 c^L + b^L \partial c^L c^L \right). \qquad (16)$$

Here we have introduced the (anti-) ghosts (b^i), c^i, corresponding to the constraints ϕ^i, fulfilling the canonical relations

$$\left\{ b^i(\sigma), c^j(\sigma') \right\}^+ = -i\delta^{ij}\delta(\sigma - \sigma'). \qquad (17)$$

The couplings are determined by the structure constants f^{ijk} of the algebra(15) according to the general prescription of [31]:

$$\mathcal{Q} = \phi^i c^i - \tfrac{1}{2} f^i_{jk} b_i c^j c^k. \qquad (18)$$

The classical nilpotency, $\mathcal{Q}^2 = 0$, is guaranteed by construction. Whether this survives in the quantum theory is the topic of the rest of this article.

2 The quantum theory

2.1 The Vacuum

We define the matter part of the vacuum $|0\rangle_p$ by the condition that

$$P^M(\xi)|0\rangle_p = 0 \ \ \forall M. \qquad (19)$$

In terms of their Fourier components this reads

$$p^M_n |0\rangle_p = 0 \ \ \forall M, n, \qquad (20)$$

which because of the commutation relations implies

$$x^M_n |0\rangle_p \neq 0 \ \ \forall M, n. \qquad (21)$$

We have arrived at these definitions by a wish to keep a relation to the $T \neq 0$ Hilbert space and vacuum. We have argued as follows:

From the expressions of the oscillators of the closed tensile string, with T denoting the tension,

$$\begin{aligned} \alpha_n^M(T) &= -in\sqrt{T}x_n^M + \frac{1}{2\sqrt{T}}p_n^M \\ \tilde{\alpha}_{-n}^M(T) &= in\sqrt{T}x_n^M + \frac{1}{2\sqrt{T}}p_n^M, \end{aligned}$$

and the requirement on the tensile vacuum

$$\alpha_n^M(T)|0\rangle_T = \tilde{\alpha}_n^M(T)|0\rangle_T = 0 \qquad \forall n > 0,$$

we find that

$$\left(-i|n|\sqrt{T}x_n^M + \frac{1}{2\sqrt{T}}p_n^M\right)|0\rangle_T = 0 \qquad \forall n \neq 0. \tag{22}$$

Should we assume a similar equation to hold also for a T independent tensionless vacuum we would be forced to choose $x_n^M|0\rangle = p_n^M|0\rangle = 0$ for all M and $n \neq 0$. This is inconsistent with the commutation relations. A possible modification of this is that only the positive modes annihilate the vacuum. However, this corresponds to the $T \to 0$ limit of a tensile theory with left and right oscillators treated differently. As we have no reason to suspect such a breaking of symmetry, we now turn to the one remaining choice, advocated in [15]. In (22) we see that the P operators become more and more important as $T \to 0$. We thus choose a vacuum, $|0\rangle_p$ which is annihilated by P operators only. Then

$$\left(-i|n|\sqrt{T}x_n^M + \frac{1}{2\sqrt{T}}p_n^M\right)|0\rangle_p \to 0 \qquad \forall n \neq 0, \tag{23}$$

when $T \to 0$ and the $|0\rangle_p$ vacuum satisfies the $T \to 0$ limit of the vacuum conditions of the tensile theory in a way consistent with the canonical commutation relations. An additional complication compared to [15] is that there are two extra dimensions. One might entertain the idea that operators acting on these dimensions should be treated differently; however, this would mean breaking of the manifest space-time conformal covariance and, since we want to examine if this symmetry is preserved in the quantized theory, this is not a convenient choice. Our final choice of the matter part of the vacuum is thus (19).

2.2 The full vacuum

Since the full theory involves ghosts we will also have to choose vacuum states for these. Our guiding principle in search of a viable ghost vacuum is that the total ghost and matter vacuum state should be physical, and thus be annihilated by the BRST charge.

To make our manipulations well defined, we have to work in a space with finite inner products. In [27] it was shown that this can be achieved by introducing an additional state space together with a well defined bilinear form. Using this formalism, we will have *bra* and *ke*t states belonging to *different* state spaces such that $\langle bra\ sector|ket\ sector\rangle = finite$ [6].

Following the prescription of [27], we will take the *ket* states to be built from our vacuum of choice, $|0\rangle_p$, and the *bra* states to be built from ${}_x\langle 0|$ satisfying ${}_x\langle 0|0\rangle_p = 1$. Since the theory contains ghosts, we will also have to consider vacuum states for these, so that the *ket* ghost vacuum is given by $|G\rangle$ and the *bra* ghost vacuum is given by $\langle G'|$, satisfying $\langle G'|G\rangle = 1$. From our choice of vacuum, and from the requirement that the BRST charge (16) should annihilate the vacuum, we find that also $\{\mathcal{Q}, P^M\} = 2iX^M c^1 + iP^M c^0 - i\partial(P^M c^L)$ must annihilate the vacuum. Commutation relations tells us that X^M cannot annihilate the vacuum. Therefore one has to impose $c^1|G\rangle = 0$. This means that $\langle G'|b^1 = 0$. Similarly we find that $\langle G'|c^{-1} = 0$ and $b^{-1}|G\rangle = 0$.

To summarize we have states built from the following vacuum:

$$|0\rangle = |0\rangle_p|G\rangle$$
$$\langle 0| = {}_x\langle 0|\langle G'|,$$

which satisfies

$$P^M|0\rangle = c^1|0\rangle = b^{-1}|0\rangle = 0$$
$$\langle 0|X^M = \langle 0|c^{-1} = \langle 0|b^1 = 0$$
$$\langle 0|0\rangle = {}_x\langle 0|0\rangle_p\langle G'|G\rangle = 1.$$

The action of c^L, b^L, c^0, b^0 on the vacuum is, for now, left undetermined.

[6] In [29], a proposal for a general BRST invariant inner product of physical states using this formalism is given. However, one should be aware that in [29] only systems with a finite number of degrees of freedom are treated. We believe that the case of infinite number of degrees of freedom can be dealt with in the same way using the regularization methods introduced in this article.

2.3 Regularization

To keep track of possible divergencies in our calculations we have to regulate. This is done as in [15], using an approximate delta function which fulfills

$$\begin{aligned}\lim_{\epsilon\rightarrow 0}\int d\sigma f(\sigma)\delta_\epsilon(\sigma) = f(0) \quad \delta_\epsilon(-\sigma) = \delta_\epsilon(\sigma)\\ \int d\sigma\delta_\epsilon(\sigma) = 1 \quad \delta_{s\epsilon}(\sigma) = \tfrac{1}{s}\delta_\epsilon(\tfrac{\sigma}{s}).\end{aligned} \tag{24}$$

In the limit $\epsilon \rightarrow 0$, the regularized delta function approaches a real Dirac delta function. The regularisation is achieved by introducing the approximate delta function in the canonical commutation relations,

$$\begin{aligned}\left[X^M(\sigma), P^N(\sigma')\right] &= i\eta^{MN}\delta_\epsilon(\sigma-\sigma')\\ \{\{b(\sigma), c(\sigma')\} &= \delta_\epsilon(\sigma-\sigma'),\end{aligned} \tag{25}$$

which amounts to adding terms which vanish as $\epsilon \rightarrow 0$ to the commutator. This formalism will allow us to isolate infinities appearing in our calculations, they will come out as terms diverging as we let ϵ go to zero.

We want to investigate if there are anomalies in the quantized version of the constraints (15) and also if there are quantum obstructions to the nilpotency of the quantum BRST charge.

To this end we have to calculate commutators of composite operators and use that the fundamental fields satisfy (25). The result of these computations is in general not a local object but it may be reinterpreted as an ϵ expansion in local quantities.

In doing this we expect to uncover possible infinities, i.e., terms proportional to $\frac{1}{\epsilon}$. It is thus crucial to control the ϵ dependence in these calculations. Before giving our prescription for obtaining this control, we illustrate the situation by way of two examples.

Consider first the distributional equivalence

$$\begin{aligned}A(\sigma)\delta_\epsilon(\sigma-\sigma') &= A(\sigma')\delta_\epsilon(\sigma-\sigma') \qquad (26)\\ &+\frac{b\epsilon^2}{2}(A''(\sigma')\delta_\epsilon(\sigma-\sigma') - 2A'(\sigma')\delta'_\epsilon(\sigma-\sigma')) + \mathcal{O}(\epsilon^4),\end{aligned}$$

where b is a constant

$$b = \int d\sigma\delta_1(\sigma)\sigma^2. \tag{27}$$

To verify this equivalence one has to use test functions and integrate with respect to σ and σ', and then use the scaling properties of the regularized delta function to bring out all ϵ corrections explicitly. Note that for the relation (26) to be a true ϵ expansion, we have to require that A and the derivatives of A are well behaved when $\epsilon \to 0$ such that there are no hidden divergencies from these fields.

Our constraints (13) are composite operators upon quantization. As usual this leads to ordering ambiguities. This has two consequences: The regularized expressions for the constraints may contain $\mathcal{O}(\epsilon)$ terms and a reordering of the fundamental fields in the constraints may generate $\frac{1}{\epsilon}$ terms. In the language of the above example (26) the A's and derivatives of A's are *not* well behaved. This may lead to problems, as is spelled out in more detail in our second example:

A relation similar to (26), but involving composite operators, is

$$\begin{aligned} 2X(\sigma)\cdot P(\sigma')\delta_\epsilon(\sigma-\sigma') &= \{X(\sigma)\cdot P(\sigma)+X(\sigma')\cdot P(\sigma')\}\,\delta_\epsilon(\sigma-\sigma') \\ &-\frac{b\epsilon^2}{2}\{X'(\sigma)\cdot P'(\sigma)+X'(\sigma')\cdot P'(\sigma')\}\,\delta_\epsilon(\sigma-\sigma') \\ &+\frac{b\epsilon^2}{2}\{X(\sigma)\cdot P'(\sigma)+X(\sigma')\cdot P'(\sigma') \\ &-X'(\sigma)\cdot P(\sigma)-X'(\sigma')\cdot P(\sigma')\}\,\delta'_\epsilon(\sigma-\sigma')+\mathcal{O}(\epsilon^4). \end{aligned} \tag{28}$$

The extra terms appear to vanish when ϵ goes to zero and we are left with the usual relation. However, if we reorder all terms, from XP ordering to PX ordering we get $\frac{1}{\epsilon}$ contributions from *all* terms since the commutator of X and P evaluated at the same point is

$$\left[X^M(\sigma),P^N(\sigma)\right]=i\eta^{MN}\delta_\epsilon(0)=i\eta^{MN}\frac{1}{\epsilon}\delta_1(0), \tag{29}$$

and each derivative of X or P will bring out an extra $\pm\frac{1}{\epsilon}$.

The two examples above reveal the problems we are faced with in trying to control the ϵ dependence: A reordering may give non-trivial corrections and we have to be very careful in choosing the ordering, not to have a hidden ϵ dependence. As illustrated in the first example, we may avoid such a dependence if all functionals of X and P are bounded in the limit $\epsilon \to 0$. We define such a bounded operator, with all X operators and their derivatives

appearing to the left of all P operators and their derivatives, to be *physically ordered*. Then the space of all states with smooth momentum dependence can be handled by studying ϵ-dependence as above. From the definition of the full vacuum we also find that physical ordering will mean that all b^1 operators appear to the left of all c^1 operators and that all c^{-1} operators appear to the left of all b^{-1} operators. The ordering of the c^0, b^0, c^L, b^L ghosts is not determined since we have not determined the ghost vacuum states corresponding to these fields. This is very similar what one does for the ordinary string. There one orders all operators with positive modes to the right of negative modes, to make sure that they annihilate the vacuum. In our case however, we are forced to choose a different vacuum, which in turn forces us to the present construction.

Our scheme for keeping track of the ϵ dependence is thus as follows; We start from the ordinary hermitean expressions and then use physical ordering in our calculations. We perform the calculations with a nonzero ϵ which allows us to take care of possible ordering constants in a consistent way. At the end, we let the regularization parameter go to zero.

It turns out that only for $D = 2$ is the quantum theory thus obtained well defined.

2.4 The BRST anomaly

In this subsection we calculate $\mathcal{Q}^2$ in the quantum theory using the operator quantization procedure described above.

In the quantum theory $\mathcal{Q}|phys\rangle = 0$ and $\langle phys|\mathcal{Q} = 0$. In particular, these equations hold true for the vacuum states. To make these equations well defined we have to physically order the BRST charge. To make sure that the BRST charge is hermitean we start from $\mathcal{Q}_H = \frac{1}{2}(\mathcal{Q} + \mathcal{Q}^\dagger)$, putting all terms in physical order, using the regularization introduced before. The full hermitean $\mathcal{Q}$, including the $\frac{1}{\epsilon}$ corrections from reordering, reads

$$\begin{aligned}\mathcal{Q}_P \;=\; & \int d\sigma \Big(P^2 c^{-1} + X \cdot P c^0 + X^2 c^1 + X' \cdot P c^L - 4ic^{-1} b^0 c^1 + 2ib^1 c^0 c^1 \\ & + 2ic^{-1} c^0 b^{-1} + i\partial b^1 c^L c^1 + i\partial c^{-1} c^L b^{-1} + ib^0 \partial c^0 c^L \\ & + ib^L \partial c^L c^L + \frac{(2-D)ia}{2\epsilon} c^0 + \frac{ia}{2\epsilon} \partial c^L + \mathcal{O}(\epsilon) \Big) ,\end{aligned} \tag{30}$$

$$a \equiv \delta_1(0),$$

where the choice of ordering for the last two cubic ghost terms is left undetermined. We observe that the last ordering correction term is a surface term and we may therefore subsequently ignore it.

We now examine the nilpotency of the physically ordered BRST charge $\mathcal{Q}_P$ using $2\mathcal{Q}_P^2 = \{\mathcal{Q}_P, \mathcal{Q}_P\}$, and keeping track of all possible ϵ corrections. We find

$$2\mathcal{Q}_P^2 = \frac{(2-D)a}{\epsilon} \int d\sigma \left(4c^1 c^{-1} + c^0 \partial c^L\right) + \frac{(D-2)A}{\epsilon} \int d\sigma 4c^1 c^{-1}, \qquad (31)$$

where

$$A = \int d\mu \delta_1(\mu) \delta_1(\mu). \qquad (32)$$

This vanishes for $D = 2$, indicating the possibility of a consistent two dimensional quantum theory. In the course of the calculation we also found that the ordering of the c^0, b^0, c^L, b^L terms do not affect this result.

2.5 Central extensions and consistency with the Jacobi identities

In this subsection we investigate the consequencies of the Jacobi identities for the constraint algebra and reexamine the nilpotency of the BRST-charge using a mode expansion of the operators.

We shall consider closed strings. Letting F denote any of the coordinates $P^M(\sigma), X^M(\sigma)$ we define the Fourier modes f_m^n by the decomposition

$$F^M(\sigma) = \frac{1}{\sqrt{\pi}} \sum_{-\infty}^{+\infty} f_n^M e^{-2in\sigma} \qquad (33)$$

so that the non-vanishing Poisson brackets for the coordinate modes are

$$\{x_m^M, p_n^N\}_{P.B.} = \eta^{MN} \delta_{m+n}. \qquad (34)$$

The Fourier modes of the constraints read

$$\phi_m^{-1} = \tfrac{1}{2} \sum_{-\infty}^{+\infty} p_k \cdot p_{m-k} = 0 \qquad (35)$$

$$\phi^0_m = \tfrac{1}{2}\sum_{-\infty}^{+\infty} x_k \cdot p_{m-k} = 0 \tag{36}$$

$$\phi^1_m = \tfrac{1}{2}\sum_{-\infty}^{+\infty} x_k \cdot x_{m-k} = 0 \tag{37}$$

$$\phi^L_m = -i\sum_{-\infty}^{+\infty} k x_k \cdot p_{m-k} = 0 \tag{38}$$

Notice that we have multiplied all the constraints by the constant $\frac{\sqrt{\pi}}{2}$ to make the notation simpler. They satisfy the algebra

$$\{\phi^a_m, \phi^L_n\}_{P.B.} = -i(m+an)\phi^a_{m+n} \tag{39}$$

$$\{\phi^L_m, \phi^L_n\}_{P.B.} = -i(m-n)\phi^L_{m+n} \tag{40}$$

$$\{\phi^a_m, \phi^b_n\}_{P.B.} = (1-\delta_{ab})(a-b)\phi^{a+b}_{m+n} \tag{41}$$

and all the other brackets vanish.

The mode expansions of the ghosts associated with the constraints $\phi^a(\sigma)$ and $\phi^L(\sigma)$ satisfy the following fundamental Poisson bracket relation

$$\{c^i_m, b^j_n\}^+_{P.B.} = -i\delta_{m+n}\delta^{ij}. \tag{42}$$

Using the relations (39)-(41) we obtain the expression for the BRST charge

$$\begin{aligned} \mathcal{Q}_C &= \sum_k (\phi^1_{-k}c^1_k + \phi^0_{-k}c^0_k + \phi^{-1}_{-k}c^{-1}_k + \phi^L_{-k}c^L_k) \\ &+ \sum_{k,l} -[2ic^1_{-k}c^{-1}_{-l}b^0_{k+l} + ic^1_{-k}c^0_{-l}b^1_{k+l} - ic^{-1}_{-k}c^0_{-l}b^{-1}_{k+l} + \\ &\quad (k+l)c^1_{-k}c^L_{-l}b^1_{k+l} + (k-l)c^{-1}_{-k}c^L_{-l}b^{-1}_{k+l} + kc^0_{-k}c^L_{-l}b^0_{k+l} + \\ &\quad \tfrac{1}{2}(k-l)c^L_{-k}c^L_{-l}b^L_{k+l}]. \end{aligned} \tag{43}$$

It can be checked that this *classical* charge has the desired property

$$\{\mathcal{Q}_C, \mathcal{Q}_C\} = 0. \tag{44}$$

To quantize the system we have to replace Poisson brackets by commutators according to $i\{\;\;\}_{(P.B.)\pm} \to [\;\;]_\pm$ $(\hbar \equiv 1)$. Then (34) and (42) become

$$[x^M_m, p^N_n] = i\delta_{m+n}\eta^{MN}, \qquad [b^i_m, c^j_n] = \delta^{ij}_{m+n} \tag{45}$$

Since, in the quantum theory x_m^M, p_m^M, c_m^i and b_m^i are non-commuting operators, one must resolve ordering ambiguities in the constraints that contain products of these operators. Since x_k^M commutes with p_{m-k}^M unless $m = 0$, we see from (35)-(38) that such ambiguities arise only in the expressions for ϕ_0^0 and ϕ_0^L.

As we have no natural way of resolving these ambiguities yet, we simply define $\hat{\phi}_0^0$ and $\hat{\phi}_0^L$ to be given by some definite ordered expressions $\hat{\phi}_0^0 \equiv :\phi_0^0:$ and $\hat{\phi}_0^L \equiv :\phi_0^L:$, where the "hat" $\hat{}$ denotes an abstract operator. In the classical theory the constraints must vanish for the allowed motions of the string. Hence in the quantum theory we demand that a physical state $|phys\rangle$ satisfy the following conditions

$$
\begin{aligned}
(\hat{\phi}_0^0 - \alpha_0)\,|phys\rangle &\equiv (:\phi_0^0: - \alpha_0)\,|phys\rangle = 0 && (46)\\
(\hat{\phi}_0^L - \alpha_L)\,|phys\rangle &\equiv (:\phi_0^L: - \alpha_L)\,|phys\rangle = 0, && (47)
\end{aligned}
$$

where because of ordering ambiguities we include an ordering constant. For the definite *physical ordering* these constants take the values discussed in Appendix A.

Let us now look at the constraint algebra. The right hand sides of equation (39), for $a = 0 = m+n$, and equation (41), for $a+b = 0 = m+n$, can be expressed in terms of ϕ_0^0. But when expressing the right hand side in terms of a definite $\hat{\phi}_0^0$ we have to take ordering corrections into account. The same is true for the $\hat{\phi}_0^L$ operator. So, in the quantum case, the constraint algebra takes the form

$$
\begin{aligned}
\left[\hat{\phi}_m^1, \hat{\phi}_n^{-1}\right] &= 2i\hat{\phi}_{m+n}^0 + d_m^{1,-1}\delta_{m+n}\\
\left[\hat{\phi}_m^L, \hat{\phi}_n^L\right] &= (m-n)\hat{\phi}_{m+n}^L + d_m^{L,L}\delta_{m+n}\\
\left[\hat{\phi}_m^0, \hat{\phi}_n^L\right] &= m\hat{\phi}_{m+n}^0 + d_m^{0,L}\delta_{m+n}\\
\left[\hat{\phi}_m^0, \hat{\phi}_n^0\right] &= d_m^{0,0}\delta_{m+n}\\
\left[\hat{\phi}_m^1, \hat{\phi}_n^0\right] &= i\hat{\phi}_{m+n}^1\\
\left[\hat{\phi}_m^{-1}, \hat{\phi}_n^0\right] &= -i\hat{\phi}_{m+n}^{-1}\\
\left[\hat{\phi}_m^1, \hat{\phi}_n^L\right] &= (m+n)\hat{\phi}_{m+n}^1\\
\left[\hat{\phi}_m^{-1}, \hat{\phi}_n^L\right] &= (m-n)\hat{\phi}_{m+n}^{-1} && (48)
\end{aligned}
$$

where all the constraints are assumed to be ordered. In these relations we

have included a possible central extension for the commutator $\left[\hat{\phi}^0_m, \hat{\phi}^0_n\right]$ since this one contains the "dangerous" operator $\hat{\phi}^0_m$ twice.

The quantum version of the classical algebra (39)-(41), thus contains central extensions and can be written in a general notation

$$\left[\hat{\Psi}^a, \hat{\Psi}^b\right] = U^{ab}_c \hat{\Psi}^c + d^{ab}. \tag{49}$$

The values of additional structure constants d^{ab} are constrained by the Jacobi identities

$$\left[\left[\hat{\Psi}^a, \hat{\Psi}^b\right], \hat{\Psi}^c\right] + \left[\left[\hat{\Psi}^c, \hat{\Psi}^a\right], \hat{\Psi}^b\right] + \left[\left[\hat{\Psi}^b, \hat{\Psi}^c\right], \hat{\Psi}^a\right] = 0 \tag{50}$$

and the commutator relation

$$\left[\hat{\Psi}^a, \hat{\Psi}^b\right] = -\left[\hat{\Psi}^b, \hat{\Psi}^a\right] \tag{51}$$

which imply that

$$U^{ab}_e d^{ec} + U^{ca}_e d^{eb} + U^{bc}_e d^{ea} = 0, \tag{52}$$

$$d^{ab} = -d^{ba}. \tag{53}$$

If we substitute the structure constants from (39)-(41) in (52) we will find that the central extensions of (49) can be written in terms of four constants $d_i, i = 1, \ldots, 4$

$$\begin{aligned}
d^{1,-1}_m &= 2(id_4 + id_3 m) && (54)\\
d^{L,L}_m &= d_1 m^3 + d_2 m && (55)\\
d^{0,L}_m &= d_3 m^2 + d_4 m && (56)\\
d^{0,0}_m &= -imd_3. && (57)
\end{aligned}$$

We assume now that $\hat{\mathcal{Q}} \equiv \mathcal{Q}_P$ is the ordered version of $\mathcal{Q}$ in (43) such that $\mathcal{Q}_P \left|phys\right\rangle = 0$. Notice that $\mathcal{Q}$ and $\mathcal{Q}_P$ are related by the following equation

$$\mathcal{Q} = \mathcal{Q}_P + A_0 c^0_0 + A_L c^L_0 \tag{58}$$

where A_0 and A_L are two ordering constants.

To check the nilpotency of $\mathcal{Q}_P$ we use the following trick [28, 26]. First we define the operators $\tilde{\phi}^i_n$ by the equation

$$\tilde{\phi}^i_n \equiv \{b^i_n, \mathcal{Q}_P\} \tag{59}$$

where $i = -1, 0, 1, L$. In the absence of anomalies the operators $\tilde{\phi}^i_n$ act as gauge generators which preserve the ghost number of the state on which they act [28, 27]. Classically one can prove that the extended constraints (59) satisfy the same algebra as the original constraints (39)-(41).

Using (43) and (59) we find that the extended constraints are given by the following relations

$$\begin{aligned}
\tilde{\phi}^{-1}_m &= :\phi^{-1}_m: + \sum_{-\infty}^{+\infty} : [2ic^1_k b^0_{m-k} + ic^0_k b^{-1}_{m-k} - (m+k) c^L_k b^{-1}_{m-k}] : && (60)\\
\tilde{\phi}^{0}_m &= :\phi^{0}_m: + \sum_{-\infty}^{+\infty} : [ic^1_k b^1_{m-k} - ic^{-1}_k b^{-1}_{m-k} - m c^L_k b^0_{m-k}] : - A_0 \delta_m && (61)\\
\tilde{\phi}^{1}_m &= :\phi^{1}_m: - \sum_{-\infty}^{+\infty} : [2ic^{-1}_k b^0_{m-k} + ic^0_k b^1_{m-k} + (m-k) c^L_k b^1_{m-k}] : && (62)\\
\tilde{\phi}^{L}_m &= :\phi^{L}_m: - \sum_{-\infty}^{+\infty} : [(k-m) c^1_k b^1_{m-k} + (k+m) c^{-1}_k b^{-1}_{m-k} + \\
&\quad + k c^0_k b^0_{m-k} + (k+m) c^L_k b^L_{m-k}] : - A_L \delta_m && (63)
\end{aligned}$$

where : : is the physical ordering defined in section 2.3.

From the relations (45) we can see that ordering ambiguities arise only in the relations for $\tilde{\phi}^0_0$ and $\tilde{\phi}^L_0$. It is instructive to define the following ghost operators

$$\begin{aligned}
\hat{G}^i_0 &= i : \sum_k c^{\hat{\imath}}_k b^{\hat{\imath}}_{-k} : && (64)\\
\hat{G}^i_L &= : \sum_k k c^{\hat{\imath}}_k b^{\hat{\imath}}_{-k} : && (65)
\end{aligned}$$

where $\hat{\imath}$ means that these indices are not summed over. Since there are ordering ambiguities the action of these operators on the vacuum will be given by the relations

$$\begin{aligned}
(\hat{G}^i_0 - \beta^i_0)\,|G\rangle = 0, && (66)\\
(\hat{G}^i_L - \beta^i_L)\,|G\rangle = 0. && (67)
\end{aligned}$$

where $\beta^i_{0,L}$ are ordering constants. This is allowed since the operators $\hat{G}^i_0$ and $\hat{G}^i_L$ commute with each other for any i. Notice that these equations correspond to setting $A_0 = \alpha_0 + \beta^1_0 - \beta^{-1}_0$ and $A_L = \alpha_L - \beta^1_L - \beta^0_L - \beta^{-1}_L - \beta^L_L$ in (61) and (63) respectively.

Since classically the extended constraints satisfy the same algebra as the original constraints, the structure constants for the extended constraint algebra are the same as the ones of the original algebra. This, combined with the fact that the only operators for which we have ordering ambiguities are again $\tilde{\phi}^0_0$ and $\tilde{\phi}^L_0$ means, that we can use the results obtained previously to write

$$
\begin{aligned}
\left[\tilde{\phi}^1_m, \tilde{\phi}^{-1}_n\right] &= 2i\tilde{\phi}^0_{m+n} + 2(i\tilde{d}_4 + i\tilde{d}_3 m)\delta_{m+n} && (68)\\
\left[\tilde{\phi}^L_m, \tilde{\phi}^L_n\right] &= (m-n)\tilde{\phi}^L_{m+n} + (\tilde{d}_1 m^3 + \tilde{d}_2 m)\delta_{m+n} && (69)\\
\left[\tilde{\phi}^0_m, \tilde{\phi}^L_n\right] &= m\tilde{\phi}^0_{m+n} + (\tilde{d}_3 m^2 + \tilde{d}_4 m)\delta_{m+n} && (70)\\
\left[\tilde{\phi}^0_m, \tilde{\phi}^0_n\right] &= -im\tilde{d}_3\delta_{m+n} && (71)\\
\left[\tilde{\phi}^1_m, \tilde{\phi}^0_n\right] &= i\tilde{\phi}^1_{m+n} && (72)\\
\left[\tilde{\phi}^{-1}_m, \tilde{\phi}^0_n\right] &= -i\tilde{\phi}^{-1}_{m+n} && (73)\\
\left[\tilde{\phi}^1_m, \tilde{\phi}^L_n\right] &= (m+n)\tilde{\phi}^1_{m+n} && (74)\\
\left[\tilde{\phi}^{-1}_m, \tilde{\phi}^L_n\right] &= (m-n)\tilde{\phi}^{-1}_{m+n}. && (75)
\end{aligned}
$$

We combine all these equations in one, writing

$$[\tilde{\phi}^i_m, \tilde{\phi}^j_n] = \sum_s \sum_{k=-\infty}^{+\infty} U^{ij}_s(m,n,k)\tilde{\phi}^s_k + \tilde{d}^{ij}_m \delta_{m+n} \qquad (76)$$

where $i, j, s = -1, 0, 1, L$.

We can now calculate the BRST anomaly using a method described in [28, 26]. There it is shown that

$$\mathcal{Q}^2_P = \tfrac{1}{2}\sum_{i,j} \tilde{d}^{ij} c^i_m c^j_{-m}. \qquad (77)$$

So, if we substitute the values of the $\tilde{d}^{ij}$'s from (68)-(75) we will have that

$$\mathcal{Q}^2_P \;=\; \tilde{d}_1 \sum_m \frac{m^3}{2} c^L_m c^L_{-m}$$

$$
\begin{aligned}
&+ \quad \tilde{d}_2 \sum_m \frac{m}{2} c^L_m c^L_{-m} \\
&+ \quad \tilde{d}_3 \sum_m (-i\frac{m}{2} c^0_m c^0_{-m} + m^2 c^0_m c^L_{-m} + 2imc^1_m c^{-1}_{-m}) \\
&+ \quad \tilde{d}_4 \sum_m (mc^0_m c^L_{-m} + 2ic^1_m c^{-1}_{-m}).
\end{aligned} \tag{78}
$$

The exact values of $\tilde{d}_f, f = 1 \ldots 4$ depend on the vacuum and ordering we have used.

The simplest and safest method to determine these constants is to calculate the matrix element of the commutators (68)-(75) between the bra and ket vacuum. From the relations (60)-(63) we can see that the ordered extended constraints satisfy the relations

$$
\tilde{\phi}^{-1}_m |0\rangle = \langle 0| \tilde{\phi}^1_m = 0 \quad , \forall m \tag{79}
$$

$$
\tilde{\phi}^0_m |0\rangle = \tilde{\phi}^L_m |0\rangle = 0 \quad , \forall m \neq 0. \tag{80}
$$

As mentioned in section 2.4 we start from the hermitean $\mathcal{Q}_H$. Then by putting all terms in physical order we can calculate the constants α_0, α_L, β^i_0 and β^i_L for this particular ordering. The calculation can be found in the Appendix A.

The expectation value of the commutator (69) is

$$
\begin{aligned}
\langle 0| [\tilde{\phi}^L_m, \tilde{\phi}^L_{-m}] |0\rangle &= 2m \langle 0| \tilde{\phi}^L_0 |0\rangle + \tilde{d}_1 m^3 + \tilde{d}_2 m \\
\Rightarrow 0 &= 2m(\alpha_L - \beta^1_L - \beta^0_L - \beta^{-1}_L - \beta^L_L) - \tilde{d}_1 m^3 + \tilde{d}_2 m \\
\Rightarrow \tilde{d}_1 = 0, \qquad \tilde{d}_2 &= -2(\alpha_L - \beta^1_L - \beta^0_L - \beta^{-1}_L - \beta^L_L).
\end{aligned} \tag{81}
$$

In the same way from (70) we have

$$
\begin{aligned}
\langle 0| [\tilde{\phi}^0_m, \tilde{\phi}^L_{-m}] |0\rangle &= m \langle 0| \tilde{\phi}^0_0 |0\rangle + \tilde{d}_3 m^2 + \tilde{d}_4 m \\
\Rightarrow 0 &= m(\alpha_0 + \beta^1_0 - \beta^{-1}_0) + \tilde{d}_3 m^2 + \tilde{d}_4 m = 0 \\
\Rightarrow \tilde{d}_3 = 0, \qquad \tilde{d}_4 &= \beta^{-1}_0 - \beta^1_0 - \alpha_0.
\end{aligned} \tag{82}
$$

So using the results from the appendix A, in (81) and (82), we have that

$$
\tilde{d}_1 = 0, \qquad \tilde{d}_2 = (D+4) \lim_{N \to +\infty} \left(\sum_{k=-N}^{+N} k \right) \tag{83}
$$

$$
\tilde{d}_3 = 0, \qquad \tilde{d}_4 = \frac{i}{4}(D-2) \lim_{N \to +\infty} \left(\sum_{k=-N}^{+N} 1 \right). \tag{84}
$$

On the other hand when we calculate the expectation value of the commutator (68) we find the following

$$
\begin{aligned}
\tilde{d}_m^{1,-1} + 2i \langle 0| \tilde{\phi}_0^0 |0\rangle &= \langle 0| [\tilde{\phi}_m^1, \tilde{\phi}_{-m}^{-1}] |0\rangle = - \langle 0| \tilde{\phi}_0^{-1} \tilde{\phi}_0^1 |0\rangle \\
&= - \langle 0| \phi_{-m}^{-1} \phi_m^1 |0\rangle - 2 \sum_{k,l} \langle 0| c_{-k-m}^1 b_k^0 c_{-l+m}^0 b_l^1 |0\rangle \\
&= \frac{1}{2}(D+2) \lim_{N\to+\infty} \left(\sum_{k=-N}^{+N} 1 \right) - 2 \lim_{N\to+\infty} \left(\sum_{k=-N}^{+N} 1 \right) \Rightarrow \\
\tilde{d}_m^{1,-1} &= 0. \qquad (85)
\end{aligned}
$$

But $\tilde{d}_m^{1,-1}$, $\tilde{d}_3$ and $\tilde{d}_4$ have to satisfy the requirement (54) which comes from the Jacobi identities. This means that the following relation should be satisfied

$$
0 = -\frac{1}{2}(D-2) \lim_{N\to+\infty} \left(\sum_{k=-N}^{+N} 1 \right). \qquad (86)
$$

This holds *only when* $D = 2$.

Substituting the results (83),(84) and (86) in (43) we obtain that

$$
\mathcal{Q}_P^2 = \tfrac{1}{2}(2+4) \lim_{N\to+\infty} \left(\sum_{k=-N}^{+N} k \right) \sum_m m c_m^L c_{-m}^L. \qquad (87)
$$

Finally using the world-sheet parity invariant regularization scheme which is discussed in the appendix we take ($\sum k \to 0$) and get

$$
\mathcal{Q}_P^2 = 0 \qquad (88)
$$

which, as shown before, holds only in the critical dimension $D = 2$.

We thus recover a consistent theory in the extended phase space of ghost and matter fields, in two space time dimensions. So we see that there are obstructions to preserve the conformal symmetry of the classical tensionless string at the quantum level in any other dimension, in agreement with the result in [15].

It is interesting to observe that the ghosts are vital for the theory to be independent of ordering prescription. This is illustrated in Appendix B using the usual tensile bosonic string as an exemple.

It is also possible to understand why the Jacobi identities do not hold using the local field language of the subsections 2.1-2.4. In general, to construct a quantized algebra one has to regularize ordering ambiguities. However, for a proper quantized algebra to exist, commutators of generators should be identified as linear combinations of the generators *in the limit when* $\epsilon \to 0$. Schematically this could be written

$$\left[\phi_\epsilon^a, \phi_\epsilon^b\right] = U_c^{ab}\phi_\epsilon^c + \mathcal{O}(\epsilon). \tag{89}$$

The generators ϕ_ϵ are ϵ dependent but have a well defined, non-singular, ϵ independent limit when $\epsilon \to 0$. In particular this means that they must be physically ordered. The terms which are not identifiable as generators of the algebra are all proportional to positive powers of ϵ and thus vanish when $\epsilon \to 0$. If the ϵ dependence cannot be removed, there is an obstruction to representing the quantized algebra as well-defined linear operators. Algebraically such problems show up when commuting "once more", *i.e.* in triple commutators. The test that triple commutators are algebraically consistent is the Jacobi identities. Thus, regularization problems of the present kind cause the Jacobi identities to fail.

It is also possible to understand technically how this comes about. The terms that one throws away in the present construction, *i.e.* terms that are proportional to powers of ϵ, may contribute in commutation relations. To appreciate this, recall what was said in subsection 2.3. In formula (28) we saw that changing the order of X and P fields brings out nontrivial negative powers of ϵ, the simplest example being

$$X^M(\sigma)P^N(\sigma) = P^N(\sigma)X^M(\sigma) + \frac{i}{\epsilon}\eta^{MN}\delta_1(0). \tag{90}$$

With this in mind, one may check that there exist commutators between physically ordered terms, seemingly proportional to positive powers of ϵ, which give contributions in further commutators that do not vanish when ϵ goes to zero. As an illustration, consider a commutator between two typical terms arising in the calculation of the algebra

$$\left[\epsilon^2 X(\sigma)\cdot X''(\sigma), P^2(\sigma')\right] = \frac{2(D+2)}{\epsilon}\delta_1''(0)\delta_\epsilon(\sigma-\sigma') + \mathcal{O}(\epsilon). \tag{91}$$

This demonstrates that throwing away $\mathcal{O}(\epsilon)$ terms *before* calculating the Jacobi identities means that we in general risk throwing away terms that

contribute in the $\epsilon \to 0$ limit. The $\epsilon \to 0$ gauge algebra is only consistently represented when the Jacobi identities are satisfied, *i.e.* for $D = 2$.

3 Discussion

In this section we collect some thoughts on the result presented above.

The discovery of $D = 2$ as a critical dimension in the quantum theory prompts us to take a closer look at that case also in the classical theory. We will show below that in this dimension the solutions to the equations of motion are simply massless particles.

The critical dimension $D = 2$ is furthermore compared with the result of [15] where no critical dimension was discovered.

We also list some topics for future considerations.

3.1 Two dimensions, a special case

We start from the action (11) and derive the V^α equations

$$\dot{\underline{X}} \cdot \partial \underline{X} = 0 \quad \underline{X'} \cdot \partial \underline{X} = 0. \tag{92}$$

Here spacetime vectors are underlined and

$$\partial \equiv V^\alpha \partial_\alpha. \tag{93}$$

From (92) it follows that $(\partial \underline{X})^2 = 0$, which, in $D = 2$ implies that [7] $\partial \underline{X} = \alpha(\xi) \underline{e}$ with $\underline{e}$ a vector in one of the two null directions. Substituting this information back into (92) we conclude that also $\dot{\underline{X}}$ and $\underline{X'}$ are proportional to $\underline{e}$. Denoting the proportionality functions by $\beta(\xi)$ and $\gamma(\xi)$ respectively, we find two expressions for $\underline{X}$:

$$\begin{aligned} \underline{X} &= \underline{e} \int \beta(\xi) d\tau + \underline{f}(\sigma) \\ \underline{X} &= \underline{e} \int \gamma(\xi) d\sigma + \underline{g}(\tau). \end{aligned} \tag{94}$$

Taking the σ-derivative of the first expression and comparing it to $\underline{X'} = \gamma(\xi)\underline{e}$ we determine $\underline{f}$. Thus we find:

$$\underline{X} = \underline{e} \int \gamma(\xi) d\sigma + \underline{c} \equiv \Gamma(\xi)\underline{e} + \underline{c}, \tag{95}$$

[7] In this context, we disregard the solutions $\partial \underline{X} = 0$ which carry zero momentum.

where $\underline{c}$ is a constant vector. We may now make a $2D$ world-sheet coordinate transformation $\Gamma(\xi) \to \tau$ to rewrite (95) as the expression for a massless particle:

$$\underline{X} = \tau \underline{e} + \underline{c}. \tag{96}$$

Note that, perhaps somewhat surprisingly, we arrived at (96) using the V^α equations only. The $\underline{X}$ equation,

$$\partial_\alpha (V^\alpha \partial \underline{X}) = 0, \tag{97}$$

will determine V^α instead. Using (95), the relation (97) becomes

$$\partial_\alpha (V^\alpha \partial \Gamma) \underline{e} = \underline{0}, \tag{98}$$

with (local) solution

$$\begin{aligned} V^\alpha \partial \Gamma &= \epsilon^{\alpha\beta} \partial_\beta h \\ \Leftrightarrow \epsilon_{\alpha\beta} V^\beta \partial \Gamma &= \partial_\alpha h. \end{aligned} \tag{99}$$

Here $h(\xi)$ is a scalar density which satisfies $\partial h = 0$.

Clearly the relation (96) is insensitive to an arbitrary σ-coordinate transformation $\sigma \to \tilde{\sigma}(\xi)$. This is the manner in which the residual (Virasoro) symmetry arises in two dimensions.

In the gauge used in (96) the solution (98) takes on the form

$$(V^0)^2 = h', \qquad V^1 V^0 = -\dot{h}. \tag{100}$$

As long as h' is non-zero this is equivalent to

$$V^0 = \sqrt{h'}, \qquad V^1 = -\dot{h}/\sqrt{h'}. \tag{101}$$

We see that the V^α's are determined by one field h. From the first relation in (100) it is clear that $h' \geq 0$. For a closed string and a globally defined h, we must have periodicity $h(\sigma + 2\pi) = h(\sigma)$, and consequently $h' \geq 0 \Rightarrow h' = 0$. From (100) we see that this leaves V^1 undetermined instead. In each case the indeterminacy reflects the residual gauge symmetry and does not represent physical degrees of freedom. For V^α of trivial topology conventional Hamiltonian treatment of the system described above gives the degrees of freedom of a massless particle.

3.2 Comparision to previous results

Having thus displayed the classical $2D$ structure of the theory we see how the discovery of a critical dimension is in agreement with previous results. Namely, in [15] the physical degrees of freedom that are quantized are the transversal ones. This excludes $D = 2$ from the outset and explains why no critical dimension is found there. Furthermore, in [15] it is shown that the conformal invariance of the massless *particle* survives quantization. The proof is again for $D > 2$, but it is reasonable to expect this result to carry over to $D = 2$.

It is also proper to compare our calculation to other known results to see if our methods are consistent. The first thing one thinks of is the Lorentz symmetry of the tensionless string. It is well known that, ignoring the space time conformal invariance, quantization goes through with no problems at all, in any space time dimension [23, 24]. To compare our calculation with that result, we notice that the structure of two of our constraints are exactly similar to the ordinary constraints in the tensionless theory. Therefore we will immediately be able to compare results if we, everywhere in our calculation, just disregard everything that has to do with our two extra constraints, ϕ^0, ϕ^1 and our two extra dimensions. We may of course keep our choice of vacuum, and accordingly our physical ordering, since the discussion of these matters is generally valid for tensionless strings. We find a BRST charge

$$\tilde{\mathcal{Q}}_P = \int d\sigma \left(\phi^{-1} c^{-1} + \phi^L c^L + i\partial c^{-1} c^L b^{-1} + i b^L \partial c^L c^L + \frac{ia}{2\epsilon} \partial c^L + \mathcal{O}(\epsilon) \right), \tag{102}$$

and we may check that $\tilde{\mathcal{Q}}_P^2$ is proportional to an integral of a total derivative and thus vanishes. Furthermore, there are no problems with the Jacobi identities, neither for bosonic nor extended constraints.

The other model which bears resemblance to our case and with which we would like to compare is the conformal particle [27], being essentially a model containing only the zero modes of the ϕ constraints. In this case, the quantized theory is consistent regardless of the dimensionality of the ambient space. We may thus conclude that it is the richness of the state space, a consequence of the extendedness of strings which causes the problems; in two dimensions the extendedness effectively vanishes, and in higher dimensions we have a consistent theory if we only look at the zero modes of the

constraints. This conclusion also fits with the results of [15]. We thus find that our method gives results that agree with those arrived at by other routes.

In this article we have not used the philosophy of [15], imposing restrictions on the physical state space to avoid the problems in $D \neq 2$. It would be interesting to see where this would lead and if we could recover the results that physical states should be space time diffeomorphism invariant. The natural generalization in our case would be to require that $\mathcal{Q}^2 = 0$ only on physical states. Since commutators between operators annihilating physical states should also annihilate physical states, one would by following this route derive a large number of constraints on physical states, possibly leading to the result of [15]. There is however another more difficult problem one would have to tackle in adopting this philosophy, namely the failure of the Jacobi identities to close in $D \neq 2$. This is a problem one would have to solve to be able to carry out the program pursued in [15].

3.3 Outlook

In this paper we have studied BRST-quantization of the conformal string and found obstructions to quantization except in two space-time dimensions. A novel feature of our treatment is the important role played by the Jacobi identities. We have also emphasized the necessity of chosing a correct vacuum and we have explained how to reconcile our results with previous ones in the literature. It remains to analyze the structure of the operator anomalies and see if it is again possible to view them as giving restrictions on the physical states of the theory. If that *is* possible,we must also answer the question of how this leads to the "topological state space" result of [15].

Another obvious avenue of investigation is to turn to the $T \to 0$ limit of the superstring and the spinning string. In particular, applying the techniques of the present paper to the superstring presents an interesting challenge.

It should furthermore be interesting to take a closer look at the two-dimensional case and see if it is possible to explicitly construct the quantum theory there.

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Appendix A Ordering constants

Here we calculate the constants $\alpha_0, \alpha_L, \beta_0^i$ and β_L^i. To do this we use the following regularization scheme

$$\sum_{-\infty}^{+\infty} \rightarrow \lim_{N\rightarrow+\infty} \sum_{-N}^{+N}. \tag{A.1}$$

This means that instead of calculating the infinite sums we first calculate the sums using finite limits $+N, -N$ and at the end we take the $N \rightarrow +\infty$ limit. In this way we see that

$$\left(\sum_{k=-\infty}^{+\infty} k \right) \rightarrow 0 \tag{A.2}$$

while on the other hand $\left(\sum_{k=-\infty}^{+\infty} 1 \right)$ remains divergent.

Starting from the relations (36),(38) and (46),(47) for the physical part of the extended constraints and (64)-(67) for the ghost part we use the ordering we defined in section 2.3 to find the following

$$\begin{aligned}
\hat{\phi}_0^0 \left|0\right\rangle &= \frac{1}{4} \sum (x_{-k} \cdot p_k + p_k \cdot x_{-k}) \left|0\right\rangle \\
&= \tfrac{1}{2} \sum (x_{-k} \cdot p_k) \left|0\right\rangle - \frac{i}{4}(d+2) \sum 1 \left|0\right\rangle \\
&\Rightarrow \alpha_0 = -\frac{i}{4}(d+2) \left(\sum_{k=-N}^{+N} 1 \right) && \text{(A.3)} \\
\hat{\phi}_0^L \left|0\right\rangle &= \frac{i}{2} \sum k (x_{-k} \cdot p_k - p_k \cdot x_{-k}) \left|0\right\rangle \\
&= i \sum k x_{-k} \cdot p_k \left|0\right\rangle - \tfrac{1}{2}(d+2) \sum k \left|0\right\rangle \\
&\Rightarrow \alpha_L = -\tfrac{1}{2}(d+2) \left(\sum_{k=-N}^{+N} k \right) && \text{(A.4)} \\
\hat{G}_0^1 \left|0\right\rangle &= \frac{-i}{2} \sum (b_{-k}^1 c_k^1 - c_k^1 b_{-k}^1) \left|0\right\rangle = -i \sum b_{-k}^1 c_k^1 \left|0\right\rangle + \frac{i}{2} \sum 1 \left|0\right\rangle \\
&\Rightarrow \beta_0^1 = \frac{i}{2} \left(\sum_{k=-N}^{+N} 1 \right) && \text{(A.5)}
\end{aligned}$$

$$\begin{aligned}
\hat{G}_0^{i\neq 1}\,|0\rangle &= \frac{i}{2}\sum(c_k^{\hat{i}}b_{-k}^{\hat{i}} - b_{-k}^{\hat{i}}c_k^{\hat{i}})\,|0\rangle = i\sum c_k^{\hat{i}}b_{-k}^{\hat{i}}\,|0\rangle - \frac{i}{2}\sum 1\,|0\rangle \\
&\Rightarrow \beta_0^{i\neq 1} = -\frac{i}{2}\left(\sum_{k=-N}^{+N} 1\right) \qquad \text{(A.6)} \\
\hat{G}_L^{1}\,|0\rangle &= -\tfrac{1}{2}\sum k(b_{-k}^{1}c_k^{1} + c_k^{1}b_{-k}^{1})\,|0\rangle = -\sum k b_{-k}^{1}c_k^{1}\,|0\rangle - \tfrac{1}{2}\sum k\,|0\rangle \\
&\Rightarrow \beta_L^{1} = -\tfrac{1}{2}\left(\sum_{k=-N}^{+N} k\right) \qquad \text{(A.7)} \\
\hat{G}_L^{i\neq 1}\,|0\rangle &= \tfrac{1}{2}\sum k(c_k^{\hat{i}}b_{-k}^{\hat{i}} + b_{-k}^{\hat{i}}c_k^{\hat{i}})\,|0\rangle = \sum k(c_k^{\hat{i}}b_{-k}^{\hat{i}})\,|0\rangle + \tfrac{1}{2}\sum k\,|0\rangle \\
&\Rightarrow \beta_L^{i\neq 1} = \tfrac{1}{2}\left(\sum_{k=-N}^{+N} k\right). \qquad \text{(A.8)}
\end{aligned}$$

We should comment here that all these constants are divergent in contrast to the usual tensile case where the ordering constants have finite values.

Appendix B Some remarks on the critical dimension

Let us calculate the Virasoro algebra for the usual tensile string using the cutoff regularization (A.1).

We have that

$$\left[\phi_m^L, \phi_{-m}^L\right] = \lim_{N\to+\infty}\begin{cases} m\alpha_0^2 + 2m\sum_{n=1}^{N-m}\alpha_{-n}\cdot\alpha_n + \frac{D}{12}m(m^2-1) \\ \qquad + \sum_{k=N-m+1}^{N}(m-k)\alpha_{-k}\cdot\alpha_k \\ \\ m\alpha_0^2 + 2m\sum_{n=1}^{N-m}\alpha_n\cdot\alpha_{-n} - \frac{D}{12}m(m^2-1) \\ \qquad + \sum_{k=N-m+1}^{N}(m-k)\alpha_k\cdot\alpha_{-k} \end{cases} \qquad \text{(B.1)}$$

where we have expressed the result in two different orderings. In the first line all the positive modes are put to the right and all negative modes to the left. The second line is ordered in the opposite way. Reordering the first line will reproduce the second.

We can take the limit $N \to +\infty$ by throwing away the last terms in eqs. (B.1). Their matrix element between states of fixed mass vanish as

$N \to +\infty$. The commutator then becomes

$$\left[\phi^L_m, \phi^L_{-m}\right] = \begin{cases} m\alpha_0^2 + 2m\sum_{n=1}^{+\infty} \alpha_{-n}\cdot\alpha_n + \frac{D}{12}m(m^2-1) \\ \\ m\alpha_0^2 + 2m\sum_{n=1}^{+\infty} \alpha_n\cdot\alpha_{-n} - \frac{D}{12}m(m^2-1) \end{cases} \tag{B.2}$$

If we now reorder the first term we will not reproduce the second. To have relations independent of the ordering prescription for $D \neq 0$ we have to introduce ghosts. Then the Virasoro operators become the extended Virasoro operators

$$\tilde{\phi}^L_m = \{Q, b_m\} = \phi^L_m + \phi^{Lc}_m = \phi^L_m + \sum_{n=-\infty}^{+\infty}(m-n)b_{m+n}c_{-n} \tag{B.3}$$

Using the relation (A.1) the algebra in the extended space becomes

$$\left[\phi^L_m, \phi^L_{-m}\right] = \lim_{N\to+\infty}\begin{cases} m\alpha_0^2 + 2m\sum_{n=1}^{N-m}\alpha_{-n}\cdot\alpha_n + \frac{D}{12}m(m^2-1) \\ +2m\sum_{k=1}^{N-m}k(b_{-k}c_k + c_{-k}b_k) + \frac{1}{6}(m-13m^3) \\ +\sum_{k=N-m+1}^{N}(m-k)\alpha_{-k}\cdot\alpha_k \\ +\sum_{k=N-m+1}^{N}(2m-k)(m+k)(c_{-k}b_k + b_{-k}c_k) \\ \\ m\alpha_0^2 + 2m\sum_{n=1}^{N-m}\alpha_n\cdot\alpha_{-n}) - \frac{D}{12}m(m^2-1) \\ +2m\sum_{k=1}^{N-m}k(b_kc_{-k} + c_kb_{-k} - \frac{1}{6}(m-13m^3) \\ +\sum_{k=N-m+1}^{N}(m-k)\alpha_k\cdot\alpha_{-k} \\ +\sum_{k=N-m+1}^{N}(2m-k)(m+k)(c_kb_{-k} + b_kc_{-k}) \end{cases} \tag{B.4}$$

In the limit $N \to +\infty$ this relation becomes

$$\left[\phi^L_m, \phi^L_{-m}\right] = \begin{cases} m\alpha_0^2 + 2m\sum_{n=1}^{+\infty}\alpha_{-n}\cdot\alpha_n + 2m\sum_{k=1}^{+\infty}k(b_{-k}c_k + c_{-k}b_k) \\ \qquad +\frac{D}{12}m(m^2-1) + \frac{1}{6}(m-13m^3) \\ \\ m\alpha_0^2 + 2m\sum_{n=1}^{+\infty}\alpha_n\cdot\alpha_{-n} + 2m\sum_{k=1}^{+\infty}k(b_kc_{-k} + c_kb_{-k}) \\ \qquad -\frac{D}{12}m(m^2-1) - \frac{1}{6}(m-13m^3) \end{cases} \tag{B.5}$$

Using ζ-function regularization $\sum_{n=1}^{+\infty} n = -\frac{1}{12}$ we can show that the two branches of (B.5) are equal only when

$$\begin{aligned} \frac{m}{6}(D-2) + \frac{1}{6}D(m^3 &- m) + \frac{1}{3}(m-13m^3) = 0 \\ \Rightarrow D &= 26 \end{aligned} \tag{B.6}$$

So independence of ordering prescription requires that $D = 26$.

We can study the conformal string in the same manner. In this case we will have

$$\left[\phi^0_m, \phi^L_{-m}\right] = \lim_{N\to+\infty} \left\{ \begin{array}{l} \frac{1}{2}m \sum_{-N+m}^{N-m} x_k \cdot p_{-k} \\ +\frac{1}{2}\sum_{N-m+1}^{N}(m-k)x_k \cdot p_{-k} \\ +\frac{1}{2}\sum_{-N}^{-N+m-1} kx_k \cdot p_{-k} \\ \\ \frac{1}{2}m \sum_{-N+m}^{N-m} p_{-k} \cdot x_k \\ +\frac{1}{2}\sum_{N-m+1}^{N}(m-k)p_{-k} \cdot x_k \\ +\frac{1}{2}\sum_{-N}^{-N+m-1} kp_{-k} \cdot x_k \end{array} \right. \tag{B.7}$$

So in the $N \to +\infty$ limit the two branches are different. Using the previous procedure we extend the space to include ghosts and we find ordering independent results only when $D = 2$.

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